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Scale invariance in turbulent front propagation

A. Pocheau
Laboratoire de Recherche en Combustion, S 252, Université de Saint Jérome, 13397 Marseille, France
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Propagation of fronts in a turbulent medium is investigated in a regime where the interaction between front and turbulence is scale invariant. The relation governing the front velocity is determined exactly, including nonlinearities, as a two-parameter family, by imposing covariance by dilatation in functional space. It differs from usual power laws because scale interaction is nonlocal in scale space, in contrast with usual systems.

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I. INTRODUCTION

Turbulence is well known to enhance transport properties of passive fields, in particular their mixing. This has been modeled by the concept of turbulent diffusion. A similar statistical problem for nonpassive fields is addressed in this paper by investigating the influence of turbulence on the propagation of autocatalytic fronts. It turns out to clarify the concept of turbulent front propagation.

Turbulent front propagation may be encountered in a number of contexts, for instance, in autocatalytic phase transition within turbulent fluids, especially turbulent combustion, or in contamination within a turbulent mixing as in epidemic propagation or in some problems of pollution. It might also be invoked in the transition to turbulence whenever it occurs through the propagation of fronts separating laminar and turbulent domains [1]. Its practical interest comes from the large enhancement of front velocity that can be easily produced with a slight level of turbulence. It especially determines engine performances in the context of turbulent combustion and the net epidemic expansion in the case of biological contamination.

Because of important technological implications, turbulent front propagation has been widely considered in combustion and we shall refer to it often in this paper. A large range of method, phenomenological [2–11], heuristic [11–17], and statistical [renormalizations [18,19], probability density function (PDF) analysis [20,21], fractal analysis [7–10]] has been applied to it but, except the renormalization approach which will turn out to contradict our results, no rigorous derivation of the turbulent velocity within well-defined hypothesis has been undertaken. This has resulted in a confused situation where a lot of incompatible laws compete without any definite criterion for validating them.

In order to clarify this situation, we look at this problem from a different side. Instead of considering it at the level of field dynamics, we focus here on an important symmetry, satisfied at least in some regimes, namely, scale invariance of the front-turbulence *interaction*, and we show that it suffices for determining the *exact* turbulent velocity by a *rigorous* analytical method [22]. At large turbulence, the front velocity then appears proportional to the turbulence intensity, independently of its normal velocity so that propagation is thoroughly supported by turbulence at the maximal possible rate. Although restricted to a scale-invariant regime, these results provide an unambiguous basis for confrontation with experiments and a promising ground for a later treatment of the corrections which should arise when scale invariance is broken.

Our paper is organized as follows. We first lay stress in Sec. II on an essential property that has to be satisfied by scale-invariant laws, covariance by dilatation, and we show through use of definite examples how to obtain covariant laws from noncovariant ones. Turning to general ground, we next solve in Sec. III for scale-invariant laws of propagation in an exact way. The specificity of front propagation with respect to scale invariance is pointed out in Sec. IV. It explains why usual power laws are not convenient here. Properties of basins of attraction with respect to dilatation in functional space are determined in Sec. V and a conclusion about our work is finally given in Sec. VI.

II. COVARIANCE BY DILATATION

In this section, special emphasis is given on an essential property of scale invariance, covariance by dilatation, which has been overlooked so far in this problem. In phase transition, where scale invariance is encountered at critical points, this property has been well known in real space for a long time and has been used to interpret the phenomenon of critical opalescence. Inspired by this analogy, we transpose it here in functional space by first stating the basis of scale analysis of front propagation, then by using two laws of turbulent combustion to point out its necessity in scale-invariant regimes and finally by showing how to apply it in the present context.

A. Scale analysis of front propagation

Let us analyze front propagation with respect to scale by looking at propagating fronts through windows W_i of various sizes L_i (Fig. 1). In scale-invariant regimes at least, we observe in each of them an effective front propagating in an effective turbulent medium. In the reference frame where this medium is globally at rest, we define the velocity $U_{T,i}$ of effective fronts as the mean front velocity

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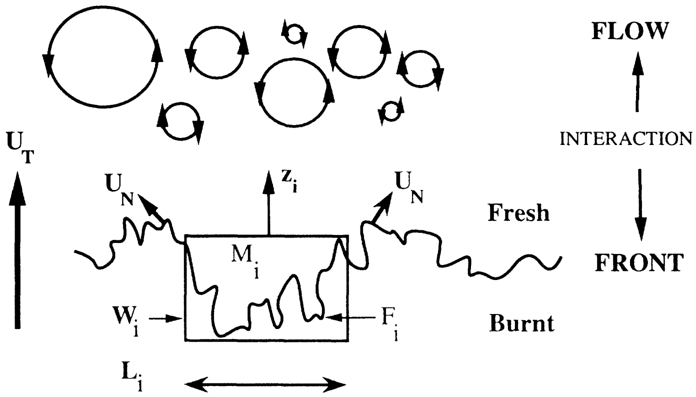

FIG. 1. Sketch of a flame front propagating in a turbulent flow. The property of scale invariance considered in this paper concerns neither the front geometry nor the turbulent flow, but the *interaction* between them. Effective fronts at scale L_i can be defined by considering front parts F_i enclosed in windows W_i of given size L_i and propagating in an effective medium M_i.

on the mean normal direction z_i in the window W_i. At the Kolmogorov scale L_K, $U_{T,i}$ is nothing but the normal velocity U_N of the front and, beyond the integral scale of turbulence L_I, it is the turbulent front velocity U_T:

$$U_{T,n}=U_T \ , \tag{1}$$

$$U_{T,0}=U_N \ , \tag{2}$$

where

$$[L_0,L_n]=[L_K,L_I] \ . \tag{3}$$

The velocities $U_{T,i}$ thus *a priori* differ from scale to scale and our goal is to determine them with respect to turbulence. Our basis assumptions will be the following: effective fronts exist at any scale; their velocity $U_{T,i}$ is independent of space and time; turbulence may be modeled by a family of statistically independent scalars (U_i') where U_i' is expected to be sufficient for describing the front-turbulence interaction in the scale range $[L_{i-1},L_i]$.

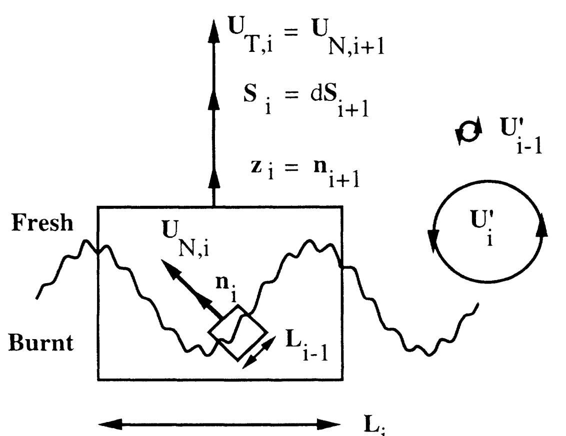

FIG. 2. Sketch of turbulent combustion analyzed at two consecutive scales. Turbulent flow is modeled by a family of statistically independent scalars U_i'. The mean direction of front propagation at scale L_i is the normal to effective fronts at scale L_{i+1}. The turbulent velocity at scale L_i is the normal velocity at scale L_{i+1}.

As detailed in Appendix A, the coherence of the description of fronts through windows enclosed one in the other as Russian dolls imposes that global properties at scale L_i behave as local ones at scale L_{i+1} (Fig. 2). In particular, the turbulent velocity at a scale L_i plays the role of normal velocity $U_{N,i+1}$ at the immediately larger scale L_{i+1}:

$$U_{T,i}=U_{N,i+1} \tag{4}$$

and the mean direction z_i of the normals n_i at scale L_i is the normal direction n_{i+1} at scale L_{i+1}:

$$z_i=n_{i+1} \ . \tag{5}$$

We emphasize that relations (4) and (5) represent an irreducible link between neighboring scales.

B. Covariance by dilatation

Let us consider two laws already proposed in studies on premixed turbulent combustion. Both of them relate the turbulent velocity U_T, the normal velocity U_N of the front, and the turbulence intensity U' [considered here as the root mean square (rms) of flow fluctuations]:

$$\frac{U_T}{U_N}=1+\frac{U'^2}{U_N^2} \ , \tag{6}$$

$$\frac{U_T}{U_N}=\exp\left[\left[\frac{U'}{U_T}\right]^2\right] \ . \tag{7}$$

The first relation has been obtained by Clavin and Williams at low-turbulence level, $U' \ll U_N$ [13]. The second, proposed by Yakhot from a renormalization procedure [19], intends to extend it to a large-turbulence regime. Both of them are derived without particularizing any scale and thus *a priori* apply to a scale-invariant regime of the front-turbulence interaction.

In scale-invariant regimes, the validity of a relation is independent of the length of the scale range in which it is applied and of its absolute position in scale space. Accordingly, the relations (6) and (7), defined on $[L_0,L_n]$, should also apply to shorter scale ranges and in particular to any range $[L_i,L_{i+1}]$ with the following correspondence:

$$U_T \rightarrow U_{T,i+1} , \tag{8}$$

$$U_N \rightarrow U_{N,i+1} = U_{T,i} , \tag{9}$$

$$U' \rightarrow U'_{i+1} . \tag{10}$$

This property provides a mean for determining any velocity increase $[U_{T,i+1}-U_{T,i}]$ and thus, by integration from L_0 to L_n, the net velocity increase $[U_T-U_N]$. However, the relation between U_T and U_N obtained this way must be the same as the original one from which it is derived (Fig. 3). This strong constraint means that scale-invariant relations must be covariant by integration in scale space. It involves the covariance by dilatation since comparing front propagation in the scale ranges $[L_i,L_{i+1}]$ and $[L_0,L_n]$ means changing the resolution of the observation by spatial dilatation or spatial contraction.

Is this criterion for scale invariance satisfied by the two above laws? Let us first consider the Clavin-Williams relation at consecutive scales:

$$\frac{U_{T,k+1}}{U_{T,k}} = 1 + \frac{U'^2_{k+1}}{U^2_{T,k}} , \tag{11}$$

$$U_{T,k} dU_{T,k} = U'^2_{k+1} dk . \tag{12}$$

Integration along the whole scale range $0 \le k < n$ yields

$$U^2_{T,n} = U^2_{T,0} + 2 \sum_{k=0}^{n-1} U'^2_{k+1} , \tag{13}$$

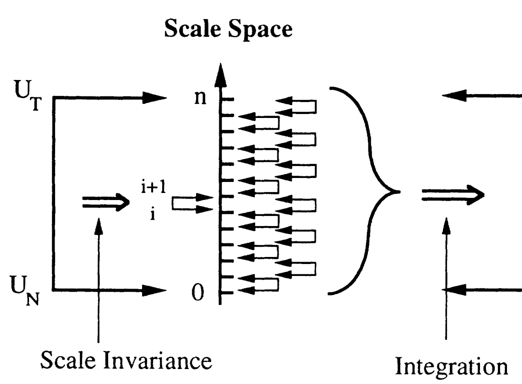

FIG. 3. Scale invariance in functional space. The axis represents length scales. Brackets symbolize relations and indicate the scale range in which they apply. In scale-invariant regimes, a relation valid in a large range $[L_0,L_n]$ must also be so on any shorter ranges, $[L_i,L_{i+1}]$, for instance. Then a new relation in $[L_0,L_n]$ may be deduced by iterative integrations of relations in elementary ranges $[L_i,L_{i+1}]$. Identifying it to the original relation from which it is derived provides a necessary requirement for scale invariance which expresses covariance by dilatation in functional space.

where root mean squares satisfy

$$U'^2 = \sum_{k=0}^{n-1} U'^2_{k+1} , \tag{14}$$

U' being the turbulence intensity in the scale range $[L_0,L_n]$, as involved in relation (6). Applying (1)–(3) and (14), we finally obtain

$$U^2_T = U^2_N + 2U'^2 . \tag{15}$$

The fact that relation (15) differs from relation (6) shows that covariance by dilatation is not satisfied. The Clavin-Williams relation thus makes a difference with respect to the length of the scale range in which it is applied: it is not scale invariant. This confirms that it is approximate and restricted to a low-turbulence level.

Let us now focus attention on the law (15) derived by renormalization of the Clavin-Williams relation and test covariance by dilatation on it. Considering it at consecutive scales:

$$U^2_{T,k+1} = U^2_{T,k} + 2U'^2_{k+1} \tag{16}$$

and integrating it along the whole scale range $0 \le k < n$, we obtain relation (13) and then, owing to (14), relation (15) again. The relation (15) thus does not depend on the length of the scale range on which it is applied so that no information on the distance between scales can be obtained by integrating it from scale to scale: it is covariant by dilatation and actually describes a scale-invariant process. Since it is equivalent to the Clavin-Williams relation at low turbulence level, it corresponds to the exact extension of this approximate law to the large-turbulence regime.

Let us apply a similar analysis to the Yakhot relation (7) by introducing the auxiliary variable $w_k = Ln(U_{T,k})$. Considering it at consecutive scales, we obtain

$$w_{k+1} - w_k = \frac{dw_k}{dk} = \frac{U'^2_{k+1}}{\exp(2w_k)} \tag{17}$$

and by integration from L_0 to L_n:

$$\exp(2w_n) = \exp(2w_0) + 2 \sum_{k=0}^{n-1} U'^2_{k+1} . \tag{18}$$

Owing to (1)–(3) and (14), this relation is equivalent to the scale-invariant relation (15) and thus different from the Yakhot relation from which it is derived.

The Yakhot relation is then not covariant by dilatation and, *a fortiori*, not suitable for describing a scale-invariant process. In contrast with relation (15), it thus fails in extending the Clavin-Williams relation to a large-turbulence regime. The origin of the disagreement with our derivation traces back to the fact that Yakhot derivation is not exact but approximate for reasons reported in Appendix B. However, looking at Yakhot relation at a coarser resolution by integrating along the scale range here too gives a scale-invariant relation which surprisingly appears to be the same as that obtained by renormalization of the Clavin-Williams relation. This coincidence will be elucidated in Sec. V.

As revealed by this simple analysis, covariance by dilatation is a subtle requirement of scale invariance capable of discriminating laws in this problem and of deriving exact relations from approximate ones. On the other hand, the fact that the scale-invariant relation (15) is simple and has been obtained by a single integration of both the above approximate laws, (6) and (7), gives the feeling that scale invariance might correspond to a simple structure in functional space. Motivated by this remark, we attack in the following section the determination of scale-invariant relations from a general viewpoint and we obtain the family of scale-invariant laws to which the particular relation (15) belongs.

III. SCALE-INVARIANT LAWS OF FRONT PROPAGATION

This section is devoted to an exact determination of the family of scale-invariant laws for turbulent front propagation. We first point out two complementary requirements of scale invariance, one of them being covariance by dilatation as emphasized in Sec. II. We next control their compatibility with well-known power laws. We then apply them to the present problem and use the corresponding constraints to select scale-invariant laws.

A. Criteria for scale invariance

Let us consider an analogy with gauge invariance in field theories for deriving two necessary conditions for scale invariance.

In classical electromagnetism, for instance, a potential V may be introduced for describing electrostatic interactions. As is well known, it is defined with respect to a reference which has only a relative meaning so that its absolute value is nonsense. Accordingly, the corresponding physical laws must be invariant by global change of V: this is the first requirement of a relevant theory. However, this is not the only one. A more subtle requirement may be obtained by noticing that, since this reference is relative, its value at different points of space should be nonsense independently. Then invariance of physical laws must be required not only in global changes of V but also in relative ones [23]. This is the second requirement. It requires the existence of a potential vector and thus of magnetism.

In this example, gauge invariance generates two requirements, one absolute and the other relative. Similar consequences may be derived from scale invariance.

In a scale-invariant system, there must be no way for obtaining information on length scales, by any means. This first implies that no absolute scale can be detected and thus that physical laws involve none of them. This is the first requirement. It is satisfied in particular by both the laws (6) and (7) since they involve no dimensional constant. However, another requirement is usually overlooked but is at least as essential: no relative values of length scales must be measurable. This implies that physical laws must be independent of the distance between the scales in terms of which they are expressed. This property is not satisfied by the laws (6) and (7) since they look different when the scale range within which they are written increases.

The first requirement corresponds to the need to destroy the concept of absolute scale in a scale-invariant system and the second to that of destroying the concept of relative scale. They refer, respectively, to covariance by absolute scale translation ($L_i \rightarrow L_{i+p}$ where p is independent of i) and to covariance by relative scale translation ($L_i \rightarrow L_{i+p}$ where p depends on i). When the former covariance is satisfied, the latter one reduces to covariance by dilatation ($L_i \rightarrow L_{ip}$).

The origin of both these requirements may also be understood within a quite usual but instructive symmetry: invariance by translation in real space. The first requirement imposes that the corresponding objects are invariant by absolute translation ($x \rightarrow x+p$ where p is independent of x) and the second that they are invariant by relative translations ($x \rightarrow x+p$ where p depends on x) (or by dilatation if the first one is satisfied). However, in this case, the first requirement is so strong (only straight objects are allowed) that it contains the second one. As shown below, the same predominancy occurs regarding scale invariance when single-variable functions are considered and this certainly explains why the second requirement is usually overlooked. It will, however, prove to be essential when multivariable functions are required, as will be shown for the present problem of front propagation.

B. Single-variable functions

In phase transition or in the Kolmogorov theory of turbulence at least, the search for scale-invariant laws is made, explicitly or implicitly, within functions of a single variable. Then, relevant variables V_i at scale L_i (turbulence intensity or correlation functions) are assumed to depend only on the scale at which they are defined, so that

$$\frac{V_i}{V_0} = f_0 \left[\frac{L_i}{L_0} \right] , \tag{19}$$

where the scale L_0 is used for adimensionalization and where f_0 is an unknown function *a priori* dependent on L_0.

Let us apply the two complementary criteria for scale invariance in order to control their compatibility with usual power laws.

Covariance by absolute scale translation ($L_i \rightarrow L_{i+p}$) implies that the variable V_i may also be written

$$\frac{V_i}{V_p} = f_p \left[\frac{L_i}{L_p} \right] , \tag{20}$$

with f_p independent of p:

$$f_p = f \ . \tag{21}$$

This means that the choice of L_0 is arbitrary and thus that the law (19) contains no absolute scale. This also implies that f satisfies

$$f(x) f(y) = f(xy) \tag{22}$$

and is thus a power law,

$$f(x)=\beta x^{\alpha} . \tag{23}$$

Covariance by dilatation ($L_i \rightarrow L_{ip}$) now requires the relation (20) to be independent of the distance $|i-p|$ between scales. This is automatically satisfied as long as f_p is independent of p, i.e., as long as the first requirement is still satisfied.

Covariance by absolute scale translation thus suffices for ensuring covariance by dilatation of single-variable functions and for selecting power laws.

C. Multivariable functions

In turbulent front propagation, velocities $U_{T,i}$ depend both on normal velocities $U_{N,i}$ and on turbulence scalars U_i'. Relation (4) then shows that the velocity $U_{T,i-1}$ is necessary for determining $U_{T,i}$ and thus, by recurrence, that $U_{T,i}$ is formally dependent on all the velocities at lower scales $U_{T,k}$, $k<i$. *A fortiori*, it is then also linked to any U_k', $k<i$ and thus implicitly to any scale L_k smaller than L_i:

$$\frac{U_{T,i}}{U_{T,0}}=f_0\left[\frac{L_i}{L_0},\frac{L_{i-1}}{L_0},\ldots,\frac{L_1}{L_0}\right] . \tag{24}$$

Single-variable functions are then forbidden to describe turbulent front propagation in scale space. In particular, simple power laws cannot be invoked in scale-invariant regimes and a new selection of laws from the two requirements of scale invariance must be performed.

D. Assumptions

Our derivation involves six assumptions, three of them having already been used in scale analysis (Sec. II A). Let us state them in the following. Since dilatation means a change of resolution in scale space, it will appear convenient to index resolutions by a superscript (r).

(a1) Effective fronts exist at any scale.

(a2) Fronts are passive: they do not modify turbulence and undergo no instability.

(a3) The normal velocity $U_{N,i}^{(r)}$ is independent of space and time.

(a4) The front-turbulence interaction in scale ranges $[L_i,L_{i+1}]$ is modeled by positive statistically independent scalars ($U_i'^{(r)}$). We stress that they may not necessarily correspond to the turbulence intensity.

(a5) The front-turbulence interaction is local in scale space so that $U_{T,i}^{(r)}$ only depends on $U_{N,i}^{(r)}$ and on $U_i'^{(r)}$. Then, owing to relation (4),

$$\frac{U_{T,i}^{(r)}}{U_{T,i-1}^{(r)}}=\Sigma_i^{(r)}\left[\frac{U_i'^{(r)}}{U_{T,i-1}^{(r)}}\right] , \tag{25}$$

where the unknown function $\Sigma_i^{(r)}$ *a priori* depends on both scales and resolution.

(a6) The function $\Sigma_i^{(r)}$ is algebraic near the origin.

Some of the properties of the function $\Sigma_i^{(r)}$ may be inferred from the expected effects of turbulence and from the assumption of passive fronts.

(p1) $\Sigma_i^{(r)}(0)=1$. Uniform flows give no velocity enhancement.

(p2) $\Sigma_i^{(r)}$ has a positive derivative. Turbulence increases front velocity.

(p3) $\Sigma_i^{(r)}$ is continuous. Since fronts are passive, they undergo no instability. Their behavior and, in particular, $\Sigma_i^{(r)}$, are regular.

(p4) $\Sigma_i^{(r)}\geq 1$ as derived from (p1) and (p2).

From these properties and assumptions, we may rewrite $\Sigma_i^{(r)}(x)$, without loss of generality, as

$$[\Sigma_i^{(r)}](x)=1+\frac{\beta}{\alpha}x^{\alpha}+x^{\alpha}R_i^{(r)}(x) \tag{26}$$

or

$$[\Sigma_i^{(r)}]^{\alpha}(x)=1+\beta x^{\alpha}+x^{\alpha}S_i^{(r)}(x) , \tag{27}$$

where $R_i^{(r)}(0)=S_i^{(r)}(0)=0$ and where α and β, positive, *a priori* depend on both i and (r): $\alpha\equiv\alpha_i^{(r)},\beta\equiv\beta_i^{(r)}$.

In this framework, we shall show that, owing to scale invariance, $\alpha_i^{(r)}$ and $\beta_i^{(r)}$ are independent of i and (r) and that $S_i^{(r)}$ vanishes. Accordingly, in scale-invariant regimes, $\Sigma_i^{(r)}$ will reduce to the following function Σ:

$$\Sigma^{\alpha}(x)=1+\beta x^{\alpha} . \tag{28}$$

E. Constraints on scale-invariant laws

Let us express the two complementary criteria for scale invariance within our framework.

Covariance by absolute scale translation implies that $\Sigma_i^{(r)}$ is independent of i:

$$\Sigma_i^{(r)}=\Sigma^{(r)} . \tag{29}$$

This only requires us to search for scale-invariant laws within the family of functions $\Sigma_i^{(r)}$ whose parameters α, β, and S are independent of i. Regarding the laws proposed in the literature in the scale range $[L_0,L_n]$ and interpreted for consecutive scales $[L_i,L_{i+1}]$, this constraint corresponds to avoiding dimensional constant. It is usually well taken into account, as for instance in both the laws (6) and (7).

Covariance by dilatation requires that $\Sigma^{(r)}$ is independent of (r):

$$\Sigma^{(r)}=\Sigma . \tag{30}$$

In contrast with single-scale functions, there is no reason for this criterion to be fulfilled as soon as the former covariance is satisfied. This constraint is thus stronger than the former one but has been overlooked so far. Let us express it in the following.

Owing to the definition (25) and (27) of $\Sigma_i^{(r)}$, our starting point is the following law between consecutive scales where, according to the covariance by absolute scale translation and by dilatation, α, β, and S are independent of k and of (r):

$$U_{T,k+1}^{(r)\alpha}=U_{T,k}^{(r)\alpha}+\beta U_{k+1}'^{(r)\alpha}+U_{k+1}'^{(r)\alpha}S(m_{k+1}^{(r)}) , \tag{31}$$

$$m_{k+1}^{(r)}=\frac{U_{k+1}'^{(r)}}{U_{T,k}^{(r)}} , \tag{32}$$

$$S(0)=0 . \tag{33}$$

A change of resolution is obtained by integrating (31) from scale L_i to scale L_j and then by redefining the scale space in order that these scales are consecutive (Fig. 4):

$$[L_i, L_j] \rightarrow [L_l, L_{l+1}] \ . \tag{34}$$

The first operation turns out to implicity remove intermediate scales L_k, $i<k<j$, from scale space and is a scale reduction. The second operation turns out to remove them explicitly and is a scale contraction. Altogether, they correspond to the following dilatation in real space:

$$x \rightarrow \frac{x}{a} \ , \quad \text{with } a = \frac{|L_j - L_i|}{|L_{l+1} - L_l|} \tag{35}$$

and yield a renormalization of laws of front propagation.

Scale reduction is performed by simply summing relation (31) from $k=i$ to $k=j-1$ and by removing terms that cancel:

$$U_{T,j}^{(r)\alpha} = U_{T,i}^{(r)\alpha} + \beta \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} + \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} S(m_{k+1}^{(r)}) \ . \tag{36}$$

Scale contraction is performed by simply redefining L_i and L_j as consecutive scales L_l and L_{l+1}. Then both the scale space and the resolution are changed, $(r) \rightarrow (r+1)$, so that we obtain the following correspondence:

$$U_{T,l}^{(r+1)} = U_{T,i}^{(r)} \ , \tag{37}$$

$$U_{T,l+1}^{(r+1)} = U_{T,j}^{(r)} \ , \tag{38}$$

and finally

$$U_{T,l+1}^{(r+1)\alpha} = U_{T,l}^{(r+1)\alpha} + \beta \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} + \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} S(m_{k+1}^{(r)}) \ . \tag{39}$$

At the new resolution $(r+1)$ and in the new scale

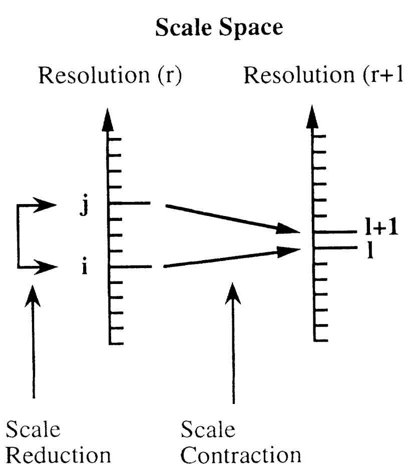

FIG. 4. Procedure for changing resolution in scale space: a scale reduction followed by a scale contraction. The first operation consists in expressing a direct relation between variables at distant scales L_i, L_j and the second one in relabeling these scales as consecutive ones.

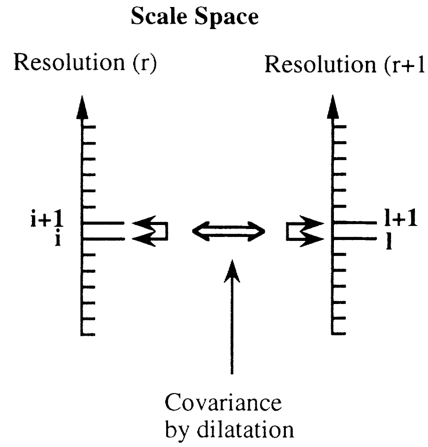

FIG. 5. Covariance by dilatation in scale space: since spatial dilatation corresponds to a change of scale resolution, covariant relations between consecutive scales must be independent of scale resolution.

space, the relation covariant to (31) may be written

$$U_{T,l+1}^{(r+1)\alpha} = U_{T,l}^{(r+1)\alpha} + \beta U_{l+1}^{\prime(r+1)\alpha} + U_{l+1}^{\prime(r+1)\alpha} S(m_{l+1}^{(r+1)}) \ . \tag{40}$$

Covariance by dilatation (Fig. 5) imposes equality between relations (39) and (40) and thus the following constraint:

$$-\beta \left[U_{l+1}^{\prime(r+1)\alpha} - \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} \right] = U_{l+1}^{\prime(r+1)\alpha} S(m_{l+1}^{(r+1)}) - \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} S(m_{k+1}^{(r)}) \ . \tag{41}$$

F. Selection of scale-invariant laws

Let us exploit the constraint (41) to select scale-invariant laws.

Since fronts are passive, turbulence is independent of the way fronts propagate and thus of front velocities. In particular, all turbulence scalars $U_k^{\prime(p)}$ are independent of the front velocity $U_{T,i}^{(r)}$. This property is *a fortiori* valid for the left-hand side of (41), and thus for its right-hand side too. Let us then take the limit $U_{T,i}^{(r)} \rightarrow \infty$ in (41) for determining their common value.

Since $\Sigma_k^{(r)} \geq 1$ (property p4), $U_{T,k}^{(r)}$ increases with k and all $U_{T,k}^{(r)}$ with $i \leq k$ tend to infinity with $U_{T,i}^{(r)}$. Then, since the turbulence scalars $U_k^{\prime(r)}$ keep constant as $U_{T,i}^{(r)}$ varies, all variables $m_{k+1}^{(r)}$ vanish in the limit considered here. Since $S(0)=0$, this implies that both sides of relation (41) are zero:

$$U_{l+1}^{\prime(r+1)\alpha} = \sum_{k=i}^{j-1} U_{k+1}^{\prime(r)\alpha} \ , \tag{42}$$

$$S(m_{l+1}^{(r+1)}) = \sum_{k=i}^{j-1} \left[\frac{U_{k+1}^{\prime(r)}}{U_{l+1}^{\prime(r+1)}} \right]^{\alpha} S(m_{k+1}^{(r)}) \ . \tag{43}$$

The relation (42) specifies the link between turbulence scalars at different resolution. It corresponds to their renormalization with respect to dilatation.

The relation (43) specifies a property of the functions S involved in scale-invariant laws. The fact that it is not satisfied by any function shows that covariance by dilatation is not included in covariance by absolute scale translation. Let us use it to select scale-invariant laws.

Constant functions S satisfy (43) owing to (42). Let us show that they are the only ones by considering both a turbulence for which no scalar $U_k'^{(r)}$ vanish and a nontrivial dilatation: $j-i>1$. We then obtain the following strict inequalities:

$$U_{k+1}'^{(r)} < U_{l+1}'^{(r+1)} \quad \text{for } i \le k < j \tag{44}$$

owing to (42), and

$$m_{k+1}^{(r)} < m_{l+1}^{(r+1)} \quad \text{for } i \le k < j \ , \tag{45}$$

since $U_{T,i}^{(r)} = U_{T,l}^{(r+1)}$ and $U_{T,k}^{(r)}$ increases with k. On the other hand, relations (42) and (43) imply

$$\min[S(m_{k+1}^{(r)})] \le S(m_{l+1}^{(r+1)}) \le \max[S(m_{k+1}^{(r)})] \quad \text{for } i \le k < j \ . \tag{46}$$

Altogether, relations (45) and (46) show that, for any positive x, S gets smaller as well as larger values than $S(x)$ in $[0,x[$:

$$\min[S(\xi)] \le S(x) \le \max[S(\xi)] \quad \text{for } \xi \in [0,x[\ . \tag{47}$$

This, of course, might yield a pathological behavior in the vicinity of the origin, unless S is a constant. In particular, continuity at the origin is sufficient for selecting constant S, as derived below.

Let us apply the property (47) to both the absolute maxima and minima of S in $[0,\infty[$. Since S is continuous (property p3), these sets of points are closed and contain their accumulation points, in particular, their minimum $x_{\min}$. Owing to (47), their values must then be the lowest possible one, $x_{\min}=0$, so that

$$\min[S(x)] = \max[S(x)] = S(0) \quad \text{for } x \in [0,\infty[\ . \tag{48}$$

The function S is thus a constant, which, owing to $S(0)=0$, vanishes: $S=0$. This selects as possible fixed points the following relations, equivalent to (28):

$$U_{T,i+1}^{(r)\alpha} = U_{T,i}^{(r)\alpha} + \beta U_{i+1}'^{(r)\alpha} \ . \tag{49}$$

Their covariance by dilatation may finally be checked straightforwardly. They correspond to the following relation between U_T, U_N, and U':

$$U_T^\alpha = U_N^\alpha + \beta U'^\alpha \ . \tag{50}$$

The scale-invariant relations (49) correspond to fixed point of renormalization with respect to dilatation. Since they involve two parameters α and β, we shall label them FP(α,β) in the remainder of the paper. We emphasize that they are obtained *exactly*. This in particular concerns their nonlinearity.

G. Turbulence scalars

In propagation laws (25), the turbulence scalars U_i' model the *interaction* between front and turbulence. *A priori*, they might then be different than the rms of the turbulent flow in contrast with what is usually implicitly assumed. For instance, they might correspond to moments of the turbulent flow of order different than two. Let us clarify them in the following.

In scale-invariant regimes, the relation (42) shows that turbulence scalars should be additive with respect to the scale range at a given power α, possibly noninteger. For $\alpha=2$, this property is satisfied by second-order moments of turbulence owing to energy conservation. If they are chosen to model turbulence in this problem, then a quadratic scale-invariant law corresponding to the fixed point FP(2,β), i.e., relation (15), is selected. Is it the single possible one or could other choices for the turbulent scalars be made in scale-invariant regimes? The answer depends on the probability density function characterizing the turbulence and on the statistical meaning of the scalars.

At first, let us consider the maximal velocity $|\mathbf{U}\cdot\mathbf{z}|_i$ of the turbulent flow $\mathbf{U}$ at scale L_i on some direction of space z. Its spatial and temporal mean satisfies relation (42) with $\alpha=1$ and thus provides an example of a choice alternative to $\alpha=2$. Let us now consider a turbulence involving a Gaussian PDF, as is natural as far as a single scalar is assumed to model turbulence. A single moment of arbitrary order α is then sufficient for describing turbulence and all other ones are proportional to it in adimensionalized forms. Then, restricting ourselves to $j=i+2$ in relation (42), the unicity of scale-invariant laws turns out to determine whether there exist different values α,α' of the orders of the turbulence moments for which both the following relations could be simultaneously satisfied:

$$U_{l+1}'^{(r+1)\alpha} = U_{i+1}'^{(r)\alpha} + U_{i+2}'^{(r)\alpha} \ , \tag{51}$$

$$U_{l+1}'^{(r+1)\alpha'} = U_{i+1}'^{(r)\alpha'} + U_{i+2}'^{(r)\alpha'} \ . \tag{52}$$

An equivalent problem consists in determining the values of α and α' for which the following equality might be satisfied for at least a positive x:

$$(x^{\alpha'}+1)^\alpha = (x^\alpha+1)^{\alpha'} \ . \tag{53}$$

The answer is that only $\alpha=\alpha'$ works and thus that a single moment of turbulence and a single scale-invariant law can model the front-turbulence interaction in this case. In particular, if $\alpha=2$ is chosen owing to energy conservation, then the only scale-invariant law is the quadratic fixed point FP(2,β).

However, if the PDF of turbulence is non-Gaussian, as is now well recognized in most turbulent flows, several independent scalars might be necessary for modeling the front-turbulence interaction, for instance, other moments of turbulence, and our derivation of scale-invariant laws should then have to be extended.

H. Asymptotic properties of scale-invariant laws

Let us consider the scale-invariant relation (50) between U_T, U_N, and U' in the limit of large turbulence, $U' \gg U_N$. Independently of the value of their parameters α and β, all of them give the same behavior:

$$U_T = O(U') \quad \text{for } U' \gg U_N \ . \tag{54}$$

Depending on the value of α, U' might not be the turbulence intensity $U'_{\rm rms}$. However, since both are proportional to the magnitude of turbulence, they should display the same asymptotic behavior:

$$U' = O(U'_{\rm rms}) \quad \text{for } U' \gg U_N \ . \tag{55}$$

Altogether, the relations (54) and (55) show that, at large-turbulence intensity, front velocity follows the turbulence intensity whatever the values of α and β are:

$$U_T = O(U'_{\rm rms}) \quad \text{for } U' \gg U_N \ . \tag{56}$$

This result corroborates a widely shared expectation [2,24]. However, it presents the advantage of being based on an exact derivation from an outstanding property of the flame-front interaction, scale invariance. It rules out relations [13–18], yielding

$$\frac{U_T}{U_N} = \left[\frac{U'}{U_N} \right]^n , \quad \text{with } n \neq 1 \quad \text{for} U' \gg U_N \tag{57}$$

or those like the Yakhot relation (7) giving logarithmic corrections [19]:

$$U_T = U' \ln \left[\frac{U_T}{U_N} \right]^{-1/2} \quad \text{for } U' \gg U_N \ . \tag{58}$$

It finally demonstrates that any experimental inflection of the ratio $U_T/U'_{\rm rms}$ at large turbulence, i.e., any experimental bending, must originate from a non-scale-invariant phenomenon.

IV. CRITICAL SURFACE FOR TURBULENT FRONT PROPAGATION

Turbulent front propagation differs from usual systems regarding scale invariance on several accounts: (i) multivariable functions are required owing to an irreducible link between consecutive scales, (ii) scale-invariant laws differ from power laws, and (iii) the search for scale invariance in turbulent combustion appears as a controversial topic whereas it usually yields a widely shared agreement in other systems.

It therefore seems that scale invariance displays an original structure in the present problem and the purpose of this section is to elucidate it. For this, we compare in the following the iterative renormalizations by dilatation of various systems, turbulent front propagation, turbulent flow, and magnetic systems (chosen here as model of phase transition), in the same kind of scale space, a functional space. Using the concepts of coherence length and of critical surface, we then exhibit a fundamental specificity of turbulent front propagation.

A. Physical systems in functional space

Let us restate the problem of turbulent front propagation in order to compare it with other systems on the same basis.

In the present study, the physical system corresponds to a process, the interaction between front and turbulence, whose effects are characterized at each scale L_i by functions Σ_i. They are thus represented by a functional P relating length scales L_i to functions Σ_i in functional space: $\Sigma_i = P[L_i]$. These functions are usually expressed in terms of physical variables (e.g., m_i) but may be considered as implicit functions of scales: $\Sigma_i = \Sigma_i(\{L_j\})$. According to this picture, turbulent front propagation appears, at each resolution, as an abstract object $P = \{L_i, L_j, P[L_i](L_j)\}$.

This description parallels that performed in real space for well-known statistical systems, the spatial coordinates being replaced by length scales. For instance, in magnetic systems, the analogs of length scales L_i, functions at scale L_i, Σ_i, and physical processes creating them, P, are, respectively, position x, spins $\mathbf{s}(x)$, and Hamiltonian H:

$$(x, \mathbf{s}(x), H) \equiv (L_i, \Sigma_i, P) \ . \tag{59}$$

However, an important difference stands in the nature of the space in which these systems are described: scale space for turbulent fronts and real space for magnetic systems. In order to recover the same space of description, let us introduce scale variables in the latter by means of correlation functions:

$$\langle s \rangle^2 [L_i] = \langle \mathbf{s}(x) \cdot \mathbf{s}(x + L_i) \rangle \ , \tag{60}$$

where $\langle \mathbf{a} \cdot \mathbf{b} \rangle$ denotes a space average of the scalar product $\mathbf{a} \cdot \mathbf{b}$. Then magnetic systems may be represented in scale space as abstract objects $H \equiv \{L_i, L_j, \langle s \rangle [L_i](L_j)\}$ in a way similar to turbulent fronts:

$$(L_i, s_i, H) \equiv (L_i, \Sigma_i, P) \ , \tag{61}$$

with $s_i = \langle s \rangle [L_i]$.

This property may also be extended to turbulent flows by describing them directly as $T \equiv \{L_i, L_j, U'[L_i](L_j)\}$ where $U'[L_i] = U'_i$ denotes in a way analogous to (60) the pair correlation of the one-dimensional (1D) component U of the flow, i.e., its turbulence intensity at scale L_i:

$$U'^2 [L_i] = \langle U(x) U(x + L_i) \rangle \ , \tag{62}$$

$$(L_i, U'_i, T) \equiv (L_i, \Sigma_i, P) \equiv (L_i, s_i, H) \ . \tag{63}$$

It is now essential to notice that, although correlations at scale L_i *a priori* depend on any length scales in scale space in both the above systems, spin lattices or turbulent flows, they are always assumed to only depend on the scale at which they are defined. They then correspond to single-variable functions in contrast with turbulent fronts where functions depending on several scales are required. This difference refers to locality or nonlocality in scale space and may bring important consequences. In particular, let us show, by introducing the concepts of coherence length and critical surface, that the objects P, H, and T describing systems in scale space belong to opposite classes.

B. Critical surfaces

Usually, in condensed-matter physics, the coherence length ξ corresponds to the distance beyond which particles may be considered as independent, and thus to the range of effective interaction forces. In the present framework, it thus corresponds to the correlation distance on the abstract objects defining systems in scale space. However, when a dilatation is performed, the new coherence length, i.e., that observed on the same system but at the new resolution, reduces with respect to the original one:

$$x \rightarrow \frac{x}{a}\ ,\quad \xi \rightarrow \frac{\xi}{a}\ . \tag{64}$$

In particular, in scale-invariant regimes, coherence lengths must be invariant in the transformation (64), owing to covariance by dilatation. They are thus either zero or infinity: $\xi=0$ or $\xi=\infty$. Let us determine which value characterizes the three systems studied above.

In turbulent fronts, functions $\Sigma_i = P[L_i]$ depend not only on the length scale L_i at which they are defined, but also on smaller scales $L_j < L_i$, so that ξ is not zero. In particular, in scale-invariant regimes where an infinite range of scales is required, ξ is infinite [Fig. 6(a)]. On the contrary, in the remaining systems, where correlation functions $\langle s \rangle [L_i]$ and $U'[L_i]$ are assumed to only depend on the scale L_i at which they are defined, ξ vanishes [Fig. 6(b)].

Let us now label S_ξ, as usual, the sets of systems having the same coherence length in scale space (Fig. 7). The above analysis shows that the three systems considered here belong either to the critical surfaces S_∞ (turbulent fronts) or S_0 (magnetic systems, turbulent flows). Since no dilatation can make a system go from one of these critical surfaces to the other, these systems thus belong to disconnected classes.

This opposition refers in particular to the abstract objects representing these systems in scale space. In scale-invariant regimes, they must display a scale-invariant geometry. In the case of turbulent fronts, they thus correspond to fractals since ξ is infinite; in the case of magnetic systems and of turbulent flows, to scale-invariant one-dimensional Euclidean objects since ξ vanishes and, more precisely, since only power laws are then allowed, to half paraboloids ($y = x^\alpha$, $x > 0$).

C. Specificity of turbulent front propagation

Turbulent front propagation, hydrodynamic turbulence, and phase transition thus correspond to fully nonlocal ($\xi=\infty$) or local ($\xi=0$) interactions in scale space and thus to opposite physical situations [Figs. 6(a), 6(b)]. In particular, power-law dependence with respect to a single scale is forbidden in turbulent combustion in contrast with usual systems. This specificity explains the controversies that have occurred in the search for scale-invariant laws in turbulent combustion: most previous

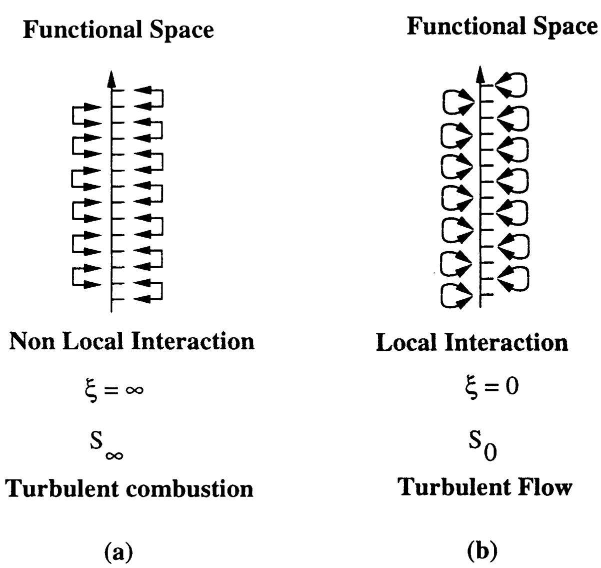

FIG. 6. Two opposite kinds of interaction in functional space. Axes represent length scales. Brackets symbolize correlations between the functions defined at each scale (e.g., $\Sigma_i = P[L_i]$) and enable us to deduce their range in scale space. (a) *Nonlocal* interaction as in turbulent combustion. The coherence length ξ of the interaction is infinite in scale space: $\xi=\infty$. These systems belong to the critical surface S_∞. (b) *Local* interaction as in usual description of turbulent flows. The coherence length ξ of the interaction vanishes in scale space: $\xi=0$. These systems belong to the critical surface S_0.

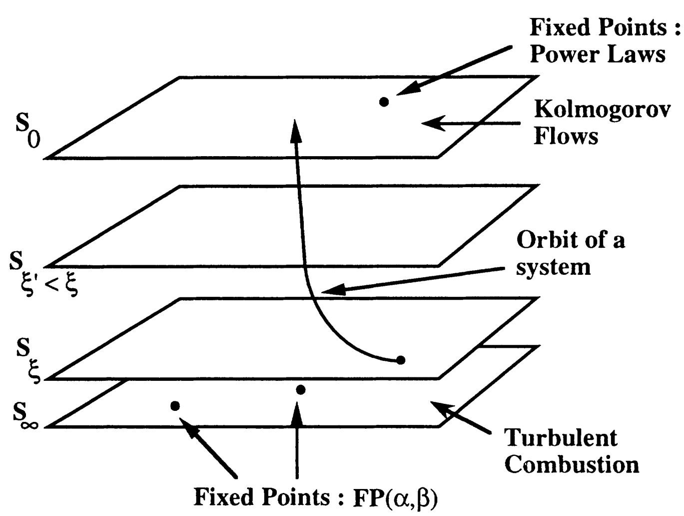

FIG. 7. Orbits of renormalization in parameter space: ξ decreases as far as renormalization proceeds [see relation (64)]. The different attractors (fixed points of S_0 and S_∞) compete to attract orbits. This legitimates the importance of fixed points in critical surface S_∞ for near critical conditions (large but finite coherence length ξ) even if they are never reached in practice. Usual systems such as Kolmogorov flows belong to S_0 but turbulent combustion belongs to S_∞. Its scale-invariant relations, the fixed points of the renormalization procedure, have then no reason to be power laws.

works, such as that of Yakhot, for instance [25], restricted themselves to usual power laws whereas another family of scale-invariant laws is required for modeling this system.

Another specificity concerns the nature of the object on which scale invariance is applied: a flow or a spin distribution in turbulence or in magnetic systems in contrast with a process in front propagation. This difference has indeed a deeper meaning than a purely formal choice of description. It refers in fact to different concepts of scale invariance: scale invariance of the process creating an object (flow cascade or front geometry), as considered in this paper, or scale invariance of these objects once they have been created. The former point of view appears more general since it enables one to determine relations whereas the latter one only solves for variables (e.g., turbulent velocity) within prescribed conditions (e.g., turbulent flow). In particular, the scale-invariant relations obtained in the present paper apply not only to scale-invariant objects but also to non-scale-invariant ones, provided that they are obtained from a scale-invariant process (e.g., turbulent fronts within non-scale-invariant flows but a scale-invariant interaction).

V. BASIN OF ATTRACTION OF SCALE-INVARIANT LAWS

In Sec. II we have observed that approximate laws may appear scale invariant when viewed at a coarser resolution. This means that, in functional space, scale-invariant laws are surrounded by basin of attraction with respect to renormalization by dilatation. This section is devoted to studying them.

For the sake of simplicity, we restrict our analysis to relations already invariant by absolute scale translation. They thus correspond to functions $\Sigma_i^{(r)}$ independent of scale L_i but not necessarily on the resolution (r):

$$\Sigma_i^{(r)}=\Sigma^{(r)}\neq\Sigma^{(r+1)} . \qquad (65)$$

Examples of such relations may be obtained, as in Sec. II, by projecting the laws of turbulent combustion which do not involve dimensional constants in the critical surface S_∞ by applying them to consecutive scales of an infinite scale space. With the same assumptions as in Sec. III, they may be written

$$\left[\frac{U_{T,k+1}^{(r)}}{U_{T,k}^{(r)}}\right]^\alpha=1+\beta m_{k+1}^{(r)\alpha}+m_{k+1}^{(r)\alpha}S^{(r)}(m_{k+1}^{(r)}) , \qquad (66)$$

$$m_{k+1}^{(r)}=\frac{U_{k+1}^{\prime(r)}}{U_{T,k}^{(r)}} , \qquad (67)$$

$$S^{(r)}(0)=0 , \qquad (68)$$

where α, β, and S *a priori* depend on (r), but not on i.

Our goal is to determine the evolution of these laws as the resolution (r) is modified by dilatation.

A. Invariant by dilatation

Let us perform dilatation in functional space in the same way as in Sec. III.

Scale reduction on the scale range $[L_i,L_j]$ gives, by summation of relation (66) from $k=i$ to $k=j-1$,

$$U_{T,j}^{(r)\alpha}=U_{T,i}^{(r)\alpha}+\beta\sum_{k=i}^{j-1}U_{k+1}^{\prime(r)\alpha}+\sum_{k=i}^{j-1}U_{k+1}^{\prime(r)\alpha}S^{(r)}(m_{k+1}^{(r)}) . \qquad (69)$$

Scale contraction simply requires the redefinition of L_i and L_j as consecutive scales L_l and L_{l+1} within the correspondence (37) and (38) between variables at resolution (r) and $(r+1)$. In addition, guided by results obtained for scale-invariant relations, we impose the same renormalization of turbulence scalars as that derived in scale-invariant regimes [relation (42)]. We then obtain

$$\left[\frac{U_{T,l+1}^{(r+1)}}{U_{T,l}^{(r+1)}}\right]^\alpha=1+\beta m_{l+1}^{(r+1)\alpha}+\sum_{k=i}^{j-1}\left[\frac{U_{k+1}^{\prime(r)}}{U_{T,l}^{(r+1)}}\right]^\alpha S^{(r)}(m_{k+1}^{(r)}) , \qquad (70)$$

$$U_{l+1}^{\prime(r+1)\alpha}=\sum_{k=i}^{j-1}U_{k+1}^{\prime(r)\alpha} . \qquad (71)$$

Relation (71) yields an upper limit for the last term of (70):

$$\left|\sum_{k=i}^{j-1}\left[\frac{U_{k+1}^{\prime(r)}}{U_{T,l}^{(r+1)}}\right]^\alpha S^{(r)}(m_{k+1}^{(r)})\right| \leq m_{l+1}^{(r+1)\alpha}\max|S^{(r)}(m_{k+1}^{(r)})| . \qquad (72)$$

Let us seek its order of expansion with respect to $m_{l+1}^{(r+1)}$.

Since $\Sigma^{(r)}\geq 1$ (property p4), $U_{T,k}^{(r)}$ increases with k. This, in addition to the facts that $U_k^{\prime(r)}\geq 0$ (assumption a4) and $U_{T,l}^{(r+1)}=U_{T,i}^{(r)}$, yields

$$0\leq m_{k+1}^{(r)}\leq m_{l+1}^{(r+1)} . \qquad (73)$$

Relations (68), (72), and (73) then show that the last term of (70) is of order $o((m_{l+1}^{(r+1)})^\alpha)$.

The renormalized function $\Sigma^{(r+1)}$ thus has the same first-order expansion as $\Sigma^{(r)}$. This means that α and β are independent of the resolution (r), even for non-scale-

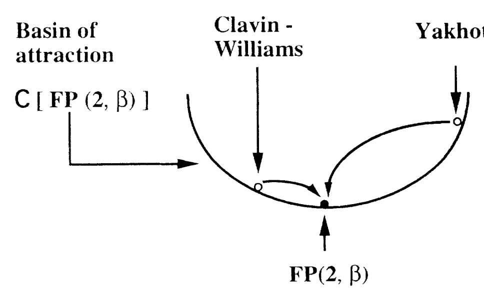

FIG. 8. Basin of attraction of turbulent combustion. A single step renormalization yields the fixed point FP(2,β) from either the Clavin-Williams (6) or the Yakhot relation (7) (see Sec. II B). This directly shows that these relations are not scale invariant. The fixed point FP(2,β) corresponds to an exact extension of the approximate relation (6) to a large-turbulence regime, in contrast to relation (7).

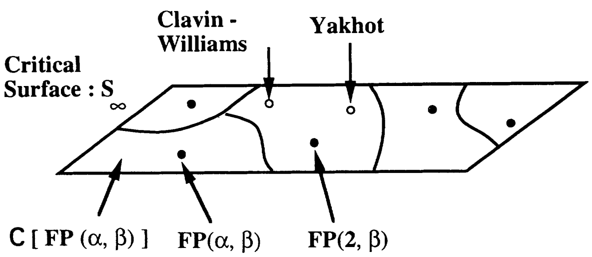

FIG. 9. Structure of the critical surface. Basins of attraction of fixed points FP(α,β) correspond to equivalence classes C[FP(α,β)] with respect to the equivalence relation $\mathfrak{T}$ linking tangent relations at the origin. The Clavin-Williams relation (6) and the Yakhot relation (7) belong to the same basin of attraction.

invariant functions and thus that the first-order expansion of Σ is an invariant of its orbit $\{\Sigma^{(n)}\}$. This gives us a useful method for obtaining the fixed point of renormalization towards which the orbit of an arbitrary function Σ might converge. Consider its first-order expansion:

$$\Sigma(x)=1+\frac{\beta}{\alpha}x^{\alpha}+o(x) . \tag{74}$$

If the orbit of Σ converges towards a fixed point, this fixed point will be FP[α,β]. This property applies in particular to relations (6) and (7) and explains the identity of their fixed point (Fig. 8).

This invariant also enables us to easily clarify the structure of the critical surface S_∞ by considering the equivalence relation $\mathfrak{T}$ according to which relations Σ are equivalent if they are tangent near the origin. Then all the orbits converging towards the same fixed point belong to the same equivalence class and basins of attraction of fixed points simply correspond to equivalence classes for $\mathfrak{T}$ (Fig. 9).

B. Orbit convergence

Is orbit convergence the rule or the exception in the critical surface? We study this question in Appendix B and we show that absolute convergence is guaranteed for very usual conditions on relations Σ and on turbulent flows.

VI. CONCLUSION

We have addressed turbulent front propagation in a scale-invariant regime. Scale-invariant laws relating front velocity, normal velocity, and turbulence have been obtained *exactly*, including their nonlinearity. They surprisingly differ from the power laws usually assumed in other scale-invariant systems. The origin of this specificity can be traced back to an irreducible link between consecutive scales in turbulent propagation, according to the fact that normal velocity is an essential ingredient of front velocity. It makes turbulent front propagation representative of a novel class of scale-invariant systems for which the interactions in scale space are *nonlocal.*

The salient features of our approach are the following. Scale invariance does not address the physical objects, flow or front, but their interaction. In contrast with usual derivations, it thus applies not to variables but to relations in functional space and then involves a wider applicability, both the front and the flow being unprescribed (possibly neither scale-invariant nor turbulent). Guided by an analogy between scale-invariant regimes of turbulent propagation and critical points of phase transition, we have looked for satisfying covariance by dilatation, a constraint which has been overlooked so far in this field. Imposing it on functional space has led us by a simple algebra to renormalized laws and to fixed points of this procedure. Skipping the complex analysis of physical fields and of front dynamics, we have then selected a two-parameter family of scale-invariant laws, analogous to power laws in usual systems:

$$U_T^{\alpha}=U_N^{\alpha}+\beta U'^{\alpha} . \tag{75}$$

In a given system, the determination of α and β may be obtained easily from expansions at a low-turbulence level. This provides an interesting means for deriving exact relations from approximate ones in this problem.

The two parameters α,β of scale-invariant laws have to be clarified in each system but, if a Gaussian statistics for turbulence is assumed, a quadratic relation should be satisfied. Anyway, independently of the value of these parameters, all laws (75) yield proportionality of front velocity to turbulence intensity in the large-turbulence limit. This result contradicts some claims that the relation between U_T and U' should depart from a straight line at large turbulence, as in relation (7), for instance [19,25]. According to the present work, this behavior, called "bending," might only originate from non-scale-invariant phenomena.

The validity of scale-invariant laws *a priori* extends to any magnitude of turbulence, provided that the actual system remains scale invariant. However, in practice, scale-invariant regimes should be bounded to some limit, indicating the occurrence of some relevant time scale in the turbulence scale range. In turbulent combustion, scale-invariant regimes are at least restricted to the so-called flamelet regime, $U_K' \ll U_N$, where U_K' is the turbulence intensity at the Kolmogorov scale and U_N the normal velocity of the front. Other causes, more intrinsic, might break scale invariance and will be investigated in detail in further studies on turbulent combustion [26].

Accurate experiments unfortunately lack for allowing detailed comparison between scale-invariant relations and actual fronts. In a recent experiment on turbulent combustion, we have sought to remove spurious contributions to front propagation in order to isolate the relevant front velocity from our data [27]. Then a good agreement has been obtained with the scale-invariant law FP(2,β). Confirmation of this result is desirable before a definitive conclusion is drawn.

In statistical physics, information about weakly non-scale-invariant regimes may be obtained by studying the vicinity of critical points. In the present system, the same approach would be worth being followed to clarify the influence of phenomena breaking scale-invariance by addressing the vicinity of fixed points in scale space. This might help to determine whether the laws governing turbulent combustion evolve continuously with respect to the magnitude of non-scale-invariant phenomena or whether they show sharp transitions. In the former case, the present theory would prove to be robust enough for describing not only scale-invariant regimes but also more realistic ones.

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APPENDIX A: SCALE ANALYSIS OF FRONT PROPAGATION

By reference to combustion, we label the medium towards which the front propagates "fresh medium" and the medium that it leaves behind "burnt medium." A scale analysis of front propagation may be performed by looking at the system at a given time through windows W_i of characteristic size L_i (Fig. 1). In each of them one observes an effective front F_i and an effective fresh medium M_i and our goal is to define their relevant variables at each scale and the relations between them.

We assume that the turbulent flow in the fresh medium may be modeled statistically by a family of vortices of size L_i represented by a family of scalars U_i' (Fig. 2). At each scale L_i, integral quantities of effective fronts F_i may be derived in a self-consistent way in windows W_i, for instance, their surface vector $\mathbf{S}_i$ or the flux of fresh medium through them, ϕ_i. On the other hand, defining local quantities raises the problem of resolution and yields a formal coupling between scales: at each scale L_i, the largest possible resolution is given by the immediately smaller scale L_{i-1} so that local variables at a scale correspond to global ones at the immediately smaller scale. Let us use this property in windows enclosed one into the other as Russian dolls for relating variables at consecutive scales (Fig. 2).

At each scale L_{i-1}, the front surface $\mathbf{S}_{i-1}$ of effective fronts corresponds to an elementary surface $d\mathbf{S}_i$ at the immediately larger scale L_i:

$$d\mathbf{S}_i=\mathbf{S}_{i-1}=\int\int_{F_{i-1}} d\mathbf{S}_{i-1} \tag{A1}$$

so that the front normal $\mathbf{n}_i$ at scale L_i may be defined as

$$\mathbf{n}_i=\frac{d\mathbf{S}_i}{dS_i} , \tag{A2}$$

where $dS_i=\|d\mathbf{S}_i\|$.

Let us now consider the problem of defining a front velocity. It turns out to relate front points at successive times. Obviously, this cannot be performed in a single way since the direction of propagation is a nonintrinsic concept dependent on front parametrization. In particular, the tangential component of front velocities is arbitrary. However, the normal component is not, since the direction of propagation is then prescribed. In any case, front velocity satisfies nevertheless the same following integral relation: the flux ϕ of front velocity over a front surface equals the volumic rate of conversion of "fresh" medium into "burnt" medium.

Let us then consider at scale L_i the normal velocities $\mathbf{U}_{N,i}$ of effective fronts with respect to fresh medium M_{i-1}. These variables are global at scale L_{i-1} and thus constant on windows W_{i-1} and on surface elements $\mathbf{S}_{i-1}=d\mathbf{S}_i$. They satisfy

$$\phi_i=\int\int_{F_i}\mathbf{U}_{N,i}d\mathbf{S}_i , \tag{A3}$$

$$\mathbf{U}_{N,i}=U_{N,i}\mathbf{n}_i . \tag{A4}$$

Let us use them for defining by the following relations, (A5) and (A6), the velocity $\mathbf{U}_{T,i}$ of effective fronts at scale L_i as the mean front velocity on the mean normal direction $\mathbf{z}_i$:

$$\mathbf{U}_{T,i}\int\int_{F_i}d\mathbf{S}_i=\int\int_{F_i}\mathbf{U}_{N,i}d\mathbf{S}_i , \tag{A5}$$

$$\mathbf{U}_{T,i}=U_{T,i}\mathbf{z}_i , \tag{A6}$$

where

$$\mathbf{z}_i=\frac{\langle\mathbf{n}_i\rangle}{\|\langle\mathbf{n}_i\rangle\|} , \tag{A7}$$

$$\langle\mathbf{n}_i\rangle=\frac{\int\int_{F_i}\mathbf{n}_i dS_i}{\int\int_{F_i}dS_i} . \tag{A8}$$

Relations (A1) and (A2) show that $\langle\mathbf{n}_i\rangle$ is collinear to $\mathbf{n}_{i+1}$ so that the mean direction of propagation at scale L_i is the normal direction at the immediately larger scale L_{i+1}:

$$\mathbf{z}_i=\mathbf{n}_{i+1} . \tag{A9}$$

Relations (A3) and (A5) show that $\mathbf{U}_{T,i}$ satisfies the integral relation of the actual front and is thus indeed a velocity of an effective front. Since it vanishes for a nonpropagating front ($U_{N,i}=0$), it is defined with respect to the mean fresh medium M_i at scale L_i. Since it is normal to the front at scale L_{i+1} and constant on windows W_i, it finally corresponds to the normal velocity of effective fronts at scale L_{i+1}:

$$\mathbf{U}_{T,i}=\mathbf{U}_{N,i+1} . \tag{A10}$$

When front velocities are assumed to be constant along the front, relations (A5) and (A10) imply

$$\Sigma_i=\frac{U_{T,i}}{U_{T,i-1}}=\frac{\int\int_{F_i}dS_i}{S_i}=R_i\geq 1 , \tag{A11}$$

where R_i is the front rugosity at scale L_i. Equality of velocity enhancement Σ_i and of rugosity R_i provides an interesting connection between front dynamics and front geometry. In particular, since rugosities are always larger than unity, the property p4 is recovered.

We emphasize that relations (A7), (A9), and (A10) provide an intrinsic link between variables at consecutive scales which makes the specificity of front propagation in scale space.

APPENDIX B: COMPARISON BETWEEN DIFFERENT RENORMALIZATION PROCEDURES AND BETWEEN FORMALLY EQUIVALENT SCALE-INVARIANT LAWS IN TURBULENT COMBUSTION

In the present paper, the fixed point FP($2,\beta$) [quadratic relation (15)] has been obtained by an exact renormalization of the Clavin-Williams relation (6) and has been explicitly shown to be scale invariant. However, in the literature of turbulent combustion, two other methods for renormalizing this relation have been applied [18,19] and two relations formally equivalent to (15) have already been proposed [5,18]. This appendix is then devoted to clarifying their difference with the approach reported in this paper and the corresponding scale-invariant relation (15).

Schelkin first proposed relation (15) in an old paper [5] but modified it some years later to a linear law [6]. Its derivation was based on the so-called "surface model" in which the flame front is considered as being made up of a lot of pockets of fresh gas surrounded by burnt gas. Introducing pockets was thought to be the only way to achieve a large relative increase of front surface, especially since Schelkin believed that a connected front could at most double its surface unless pockets were introduced. Since it is derived from a phenomenological model, the Schelkin relation is thus approximate.

Recently, Sivashinsky has obtained the following relation, formally equivalent to (15), by a single renormalization along the scale range [18]:

$$U_T = U_N \left[1+\beta \frac{U'^2}{U_N^2} \right]^{1/2} . \tag{B1}$$

Its procedure is exact but, since its starting point is an expansion of Σ at first order in $(U'/U_N)^2$, equivalent to the Clavin-Williams relation (6), its validity is *a priori* restricted to a weak turbulence regime, $U' \ll U_N$ and more precisely to a first-order expansion in $(U'/U_N)^2$. At this order, it is then simply similar to the Clavin-Williams relation (6) from which it is derived, in agreement with Sec. V A.

A priori, the remark regarding the range of validity of relation (B1) might also apply to relation (15) since it is also derived by renormalization of the approximate relation (6) in Sec. II B. Its exact nature has been proved, however, by showing that it is invariant by renormalization and thus scale invariant. On the other hand, there is no way to show the exact nature of (B1) in the framework of Sivashinsky since covariance by dilatation is not addressed.

Another procedure of renormalization of the Clavin-Williams relation (6) has been proposed by Yakhot [19] and has yielded the relation (7). Since it differs from the relation (15) obtained on the same basis, one of them at least is not exact. In particular, relations (17) and (18) show that the Yakhot relation is not invariant by renormalization and thus fails in describing a scale-invariant regime. In addition, as noticed by Sivashinsky [18], Yakhot derivation would turn out, within our framework, to divide relation (12) by $U_{T,k}^2$ and to fix $U_{T,k}$ at the value $U_{T,j}$ on the right-hand side. This renormalization procedure is thus approximate and restricted to the first order in $(U'/U_N)^2$. It yields the Yakhot relation (7) which, at this order, is similar to the Clavin-Williams relation but which, when extrapolated to large values of U'/U_N, incorrectly lowers the turbulent velocity $U_T(U_{T,k} < U_{T,j})$ and gives rise to an artificial "bending."

The relations previously proposed in the literature by Yakhot and Sivashinsky from a renormalization procedure are thus approximate for two different reasons: approximate starting point of renormalization for both Sivashinsky and Yakhot and approximate procedure in addition for Yakhot. The fact that renormalization is performed only once prevents these authors from recognizing whether a fixed point is reached (it is in fact so in Sivashinsky relation). For this reason, their relations are restricted to a weak turbulence regime, even if, as for the Sivashinsky relation, they formally look similar to the exact relation (15).

Finally, beyond a formal equivalence, the relations of Schelkin and Sivashinsky show important differences with the fixed point relation (15): the former one is a phenomenological estimate and the validity of the latter one is in fact restricted to a weakly nonlinear regime. In contrast, we emphasize that the scale-invariant relation (15) [or more generally (75)] proposed here is derived *exactly* from definite physical assumptions about the physical process (Secs. III F and V A) and that, even if it is obtained by renormalization of an approximate relation (Sec. II B), its exact nature is assessed by the fact that it is a fixed point. It should then apply in the *whole range* of turbulence intensities for which the scale-invariance assumption is valid.

APPENDIX C: ORBIT CONVERGENCE IN THE CRITICAL SURFACE S_∞

Although a lot of situations concerning orbits may occur, we want to point out that convergence towards a fixed point should be the most usual case in the critical surface. Accordingly, we do not consider the most general case but make some natural assumptions about functions S and turbulence scalars (U_i').

From Sec. V A, we know that the only fixed point towards which a function $\Sigma^{(r)}$ defined by relations (25), (31), and (32) might converge is FP(α,β) satisfying

$$\left[\frac{U_{T,i+1}}{U_{T,i}} \right]^\alpha = 1+\beta m_{i+1}^\alpha , \tag{C1}$$

where m_{i+1} is defined in (32). At each step of renormalization, a distance between renormalized functions $\Sigma^{(p)}$ and this fixed point may be obtained from the remainder function $R^{(p)}$ of the following expansions:

$$\left[\frac{U_{T,i+1}^{(r)}}{U_{T,i}^{(r)}} \right]^\alpha = 1+\beta m_{i+1}^{(r)\alpha} + R^{(p)}(m_{i+1}^{(r)}) . \tag{C2}$$

In particular, at resolutions (r) and $(r+1)$, relations (31) and (39) give, respectively,

$$R^{(r)}(m_{i+1}^{(r)})=m_{i+1}^{(r)\alpha}S^{(r)}(m_{i+1}^{(r)}) \ , \tag{C3}$$

$$R^{(r+1)}(m_{k+1}^{(r)})=\sum_{k=i}^{j-1}\left[\frac{U_{k+1}^{\prime(r)}}{U_{T,l}^{(r+1)}}\right]^{\alpha}S^{(r)}(m_{k+1}^{(r)}) \tag{C4}$$

within the correspondence (37), (38), and the scale contraction (34). Let us seek conditions for which functions $R^{(r)}$ may be shown to uniformly decrease with (r).

As a first step, let us assume that $|S^{(r)}|$ is an increasing function, in agreement with the property p2 expected for $\Sigma^{(r)}$. Then relations (C4), (72), and (73) give, within the variable renormalization (42),

$$|R^{(r+1)}(m_{l+1}^{(r+1)})|\le m_{l+1}^{(r+1)\alpha}|S^{(r)}|(m_{l+1}^{(r+1)}) \tag{C5}$$

and, by comparison with (C3),

$$|R^{(r+1)}|\le|R^{(r)}| \tag{C6}$$

so that the orbit of $\Sigma(r)$ stays in the vicinity of the fixed point FP(α,β).

Let us now assume that there exist (i) a positive exponent μ such that the function $x^{-\mu}|S^{(r)}|(x)$ is an increasing function, and (ii) a constant c, $0\le c<1$, such that, for any k, $i\le k<j$, $U_{k+1}^{\prime(r)}/U_{l+1}^{\prime(r+1)}<c<1$, the variables being renormalized according to (42).

The latter condition is satisfied, for instance, in a Kolmogorov-like turbulence, i.e., a turbulence such that the ratio $U_{k+1}^{\prime(r)}/U_{k}^{\prime(r)}$ is a constant along the scale range: $U_{k+1}^{\prime(r)}/U_{k}^{\prime(r)}=\rho$, $(U_{k}^{\prime(r)})=(\rho^{k})$ and finally, for $i\le k<j$,

$$\frac{U_{k+1}^{\prime(r)}}{U_{l+1}^{\prime(r+1)}}<\left[\frac{\rho^{\alpha}}{1+\rho^{\alpha}}\right]^{1/\alpha}=c<1 \ . \tag{C7}$$

Since $\Sigma^{(r)}\ge 1$ (property p4), the relation (73) is satisfied:

$$0\le m_{k+1}^{(r)}\le m_{l+1}^{(r+1)} \tag{C8}$$

so that the condition (i) and the relations (37) and (C7) imply

$$|S^{(r)}|(m_{k+1}^{(r)})\le c^{\mu}|S^{(r)}|(m_{l+1}^{(r+1)}) \tag{C9}$$

and finally from (C4), (C9), (42), and (C3):

$$|R^{(r+1)}|\le k|R^{(r)}| \ , \tag{C10}$$

where $k=c^{\mu}$ is strictly smaller than one: $k<1$.

This ensures absolute convergence of the orbit of $\Sigma^{(r)}$ towards the fixed point FP(α,β).

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